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FUGLEDE CONJECTURE HOLDS FOR CONVEX PLANAR DOMAINS

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Abstract. Let Ω be a compact convex domain in the plane. We prove that $L^2(\Omega)$ has an orthogonal basis of exponentials if and only if Ω tiles the plane by translation.

Section 0: Introduction

Let Ω be a domain in $\mathbb{R}^d$, i.e., Ω is a Lebesgue measurable subset of $\mathbb{R}^d$ with finite non-zero Lebesgue measure. We say that a set $\Lambda \subset \mathbb{R}^d$ is a *spectrum* of Ω if $\{e^{2\pi i x\cdot\lambda}\}_{\lambda\in\Lambda}$ is an orthogonal basis of $L^2(\Omega)$.

Fuglede Conjecture. ([Fug74]) A domain Ω admits a spectrum if and only if it is possible to tile $\mathbb{R}^d$ by a family of translates of Ω.

If a tiling set or a spectrum set is assumed to be a lattice, then the Fuglede Conjecture follows easily by the Poisson summation formula. In general, this conjecture is nowhere near resolution, even in dimension one. However, there is some recent progress under an additional assumption that Ω is convex. In [IKP99], the authors prove that the ball does not admit a spectrum in any dimension greater than one. In [Kol99], Kolountzakis proves that a non-symmetric convex body does not admit a spectrum. In [IKT00], the authors prove that any convex body in $\mathbb{R}^d$, $d > 1$, with a smooth boundary, does not admit a spectrum. In two dimensions, the same conclusion holds if the boundary is piece-wise smooth and has at least one point of non-vanishing curvature. The main result of this paper is the following:

Theorem 0.1. *Let Ω be a convex compact set in the plane. The Fuglede conjecture holds. More precisely, Ω admits a spectrum if and only if Ω is either a quadrilateral or a hexagon.*

Our task is simplified by the following result due to Kolountzakis. See [Kol99].

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Theorem 0.2. *Convex non-symmetric subsets of $\mathbb{R}^d$ do not admit a spectrum.*

Thus, it suffices to prove Theorem 0.1 for symmetric sets. Recall that a set Ω is symmetric with respect to the origin when $x \in \partial\Omega$ if and only if $-x \in \partial\Omega$.

This paper is organized as follows. The first section deals with basic properties of spectra. The second section is dedicated to the properties of the Fourier transform of the characteristic function of a convex set. In the third section we prove Theorem 0.1 for polygons, and in the fourth section we prove that any convex set which is not a polygon does not admit a spectrum, thus completing the proof of Theorem 0.1.

Section 1: Basic properties of spectra

Let

$$Z_\Omega = \left\{ \xi \in \mathbb{R}^d : \hat{\chi}_\Omega(\xi) = \int_\Omega e^{-2\pi i \xi \cdot x} \, dx = 0 \right\}. \tag{1.1}$$

The orthogonality of a spectrum Λ means precisely that

$$\lambda - \lambda' \in Z_\Omega \text{ for all } \lambda, \lambda' \in \Lambda, \lambda \neq \lambda'. \tag{1.2}$$

It follows that the points of a spectrum Λ are separated in the sense that

$$|\lambda - \lambda'| \gtrsim 1 \text{ for all } \lambda \neq \lambda', \quad \lambda, \lambda' \in \Lambda. \tag{1.3}$$

Here, and throughout the paper, $a \lesssim b$ means that there exists a positive constant C such that $a \leq Cb$. We say that $a \approx b$ if $a \lesssim b$ and $a \gtrsim b$.

The following result is due to Landau. See [Lan67]. Let

$$D_R^+ = \max_{x \in \mathbb{R}^n} \#\{\Lambda \cap Q_R(x)\}, \tag{1.4}$$

where $Q_R(x)$ is a cube of sidelength $2R$ centered at x, and let

$$D_R^- = \min_{x \in \mathbb{R}^n} \#\{\Lambda \cap Q_R(x)\}. \tag{1.5}$$

Then

$$\limsup_{R \to \infty} \frac{D_R^\pm}{(2R)^n} = |\Omega|. \tag{1.6}$$

It is at times convenient to use the following related result. We only state the special case we need for the proof of Theorem 0.1. For a more general version see [IosPed99].

Theorem 1.1. *Let Ω be a convex domain in $\mathbb{R}^2$. Then there exists a universal constant C such that if*

$$R \geq C\left(\frac{|\partial\Omega|}{|\Omega|}\right), \tag{1.7}$$

then

$$\Lambda \cap Q_R(\mu) \neq \emptyset \tag{1.8}$$

for every $\mu \in \mathbb{R}^2$, and any set Λ such that E_Λ is an exponential basis for $L^2(\Omega)$, where $Q_R(\mu)$ denotes the cube of sidelength $2R$ centered at μ.

The proof of Theorem 1.1, and the preceding result due to Landau, are not difficult. Both proofs follow, with some work, from the fact that Ω admits a spectrum Λ if and only if

$$\sum_{\Lambda} |\widehat{\chi}_\Omega(x-\lambda)|^2 \equiv 1, \tag{1.9}$$

and some averaging arguments. To say that Ω admits a spectrum Λ means that the Bessel formula $||f||^2_{L^2(\Omega)} = \sum_\Lambda |\hat{f}(\lambda)|^2$ holds. Since the exponentials are dense, it is enough to establish such a formula with $f = e^{2\pi i x\cdot\xi}$, which is precisely the formula (1.9).

Section 2: Basic properties of $\widehat{\chi}_\Omega$ and related properties of convex sets

Throughout this section, and the rest of the paper, Ω denotes a convex compact planar domain. The first two results in this section are standard and can be found in many books on harmonic analysis or convex geometry.

Lemma 2.1. $|\widehat{\chi}_\Omega(\xi)| \lesssim \frac{diam\Omega}{|\xi|}$. *Moreover, if Ω is contained in a ball of radius r centered at the origin, then $|\nabla\widehat{\chi}_\Omega(\xi)| \lesssim \frac{r^2}{|\xi|}$.*

The lemma follows from the divergence theorem which reduces the integral over Ω to the integral over $\partial\Omega$ with a factor of $\frac{1}{|\xi|}$, and the fact that convexity implies that the measure of the boundary $\partial\Omega$ is bounded by a constant multiple of the diameter. The second assertion follows similarly.

Lemma 2.2. *Suppose that ξ makes an angle of at least θ with every vector normal to the boundary of Ω. Then*

$$|\widehat{\chi}_\Omega(\xi)| \lesssim \frac{1}{\theta|\xi|^2}. \tag{2.1}$$

Moreover, if Ω is contained in a ball of radius r, then $|\nabla\widehat{\chi}_\Omega(\xi)| \lesssim \frac{r}{|\theta|\xi|^2|}$.

To prove this, one can again reduce the integral to the boundary while gaining a factor $\frac{1}{|\xi|}$. We may parameterize a piece of the boundary in the form $\{(s, -\gamma(s)+c) : a \leq s \leq b\}$,

where γ is a convex function, and, without loss of generality, $c = 0$, $a = 0$, $b = 1$, and $\gamma(0) = \gamma'(0) = 0$. We are left to compute

$$\int_0^1 e^{i(s\xi_1 - \gamma(s)\xi_2)} J(s)ds, \tag{2.2}$$

where $J(s)$ is a nice bounded function that arises in the application of the divergence theorem. The gradient of the phase function $s\xi_1 - \gamma(s)\xi_2$ is $\xi_2 \left(\frac{\xi_1}{\xi_2} - \gamma'(s)\right)$, and our assumption that ξ makes an angle of at least θ with every vector normal to the boundary of Ω means that the absolute value of this expression is bounded from below by $|\xi_2|\theta$. Integrating by parts once we complete the proof in the case $|\xi_1| \lesssim |\xi_2|$. If $|\xi_1| >> |\xi_2|$, the absolute value of the derivative of $s\xi_1 - \gamma(s)\xi_2$ is bounded below by $|\xi_1|$, so integration by parts completes the proof. The second assertion follows similarly.

Lemma 2.3. *Let f be a non-negative concave function on an interval $[-1/2, 1/2]$. Then, for every $0 < \delta \lesssim 1$, there exists $R \approx \frac{1}{\delta}$ such that $|\hat{f}(R)| \gtrsim \delta f\left(\frac{1}{2} - \delta\right)$.*

To see this, let ϕ be a positive function such that $\phi(x) \lesssim (1 + |x|)^{-2}$, $\widehat{\phi}$ is compactly supported, and $\phi(0) = 1$ in a small neighborhood of the origin. Consider

$$\int f\left(\frac{1}{2} - \delta t\right) (\phi(t+1) - K\phi(K(t+1)))dt, \tag{2.3}$$

where f is defined to be 0 outside of $[a, b]$ and K is a large positive number. If K is sufficiently large, $(\phi(t+1) - K\phi(K(t+1)))$ is positive for $t > 0$, and ≈ 1 on $[\frac{1}{2}, 1]$. It follows that

$$\int f\left(\frac{1}{2} - \delta t\right) (\phi(t+1) - K\phi(K(t+1)))dt \gtrsim f\left(\frac{1}{2} - \delta\right). \tag{2.4}$$

Taking Fourier transforms, we see that

$$\int \frac{1}{\delta} \hat{f}\left(\frac{r}{\delta}\right) e^{i\pi r} \left(\hat{\phi}(r) - \hat{\phi}\left(\frac{r}{K}\right)\right) dr \gtrsim f\left(\frac{1}{2} - \delta\right). \tag{2.5}$$

Multiplying both sides by δ and using the compact support of $\hat{\phi}(r) - \hat{\phi}\left(\frac{r}{K}\right)$, we complete the proof.

Corollary 2.4. *Let Ω be a convex body of the form*

$$\Omega = \{(x, y) : a \leq x \leq b, \ -g(x) \leq y \leq f(x)\}, \tag{2.6}$$

where f and g are non-negative concave functions on $[a, b]$. Then for every $0 < \delta \lesssim b - a$, there exists $R \approx \frac{1}{\delta}$ such that

$$|\widehat{\chi}_\Omega| \gtrsim \delta \left(f\left(\frac{1}{2} - \delta\right) + g\left(\frac{1}{2} - \delta\right)\right). \tag{2.6}$$

Section III: Lattice properties of spectra

Let Ω be a compact convex body in $\mathbf{R}^2$ which is symmetric around the origin, but is not a quadrilateral. Let Λ be a spectrum of Ω which contains the origin. The aim of this section is to prove the following two propositions which show that if a spectrum exists, it must be very lattice-like in the following sense.

Proposition 3.1. *Let I be a maximal closed interval in $\partial\Omega$ with midpoint x. Then*

$$\xi \cdot 2x \in \mathbf{Z} \tag{3.1}$$

for all $\xi \in \Lambda$.

Proposition 3.2. *Let x be an element of $\partial\Omega$ which has a unit normal n and which is not contained in any closed interval in Ω. Then*

$$\xi \cdot 2x \in \mathbf{Z} \tag{3.2}$$

for all $\xi \in \Lambda$.

In the next Section we shall show how these facts can be used to show that the only convex bodies which admit spectra are quadrilaterals and hexagons.

Proof of Proposition 3.1. We may rescale so that $x = e_1$, the coordinate direction $(1, 0 \ldots, 0)$, and I is the interval from $(e_1 - e_2)/2$ to $(e_1 + e_2)/2$. Thus, we must show that

$$\Lambda \subset \mathbf{Z} \times \mathbf{R}. \tag{3.3}$$

The set Ω thus contains the unit square $Q := [-1/2, 1/2]^2$. Since we are assuming Ω is not a quadrilateral, we therefore have $|\Omega| > 1$. In particular, Λ has asymptotic density strictly greater than 1, i.e the expression (1.6) is strictly greater than 1.

A direct computation shows that

$$\hat{\chi}_Q(\xi_1, \xi_2) = \frac{\sin(\pi\xi_1)\sin(\pi\xi_2)}{\pi^2 \xi_1 \xi_2}. \tag{3.4}$$

The zero set of this is

$$Z_Q := \{(\xi_1, \xi_2) : \xi_1 \in \mathbf{Z} - \{0\} \text{ or } \xi_2 \in \mathbf{Z} - \{0\}\}. \tag{3.5}$$

Note that $Z_Q \subset G$, where G is the Cartesian grid

$$G := (\mathbf{Z} \times \mathbf{R}) \cup (\mathbf{R} \times \mathbf{Z}). \tag{3.6}$$

Heuristically, we expect the zero set Z_Ω of $\hat{\chi}_\Omega$ to approximate Z_Q in the region $|\xi_1| \gg |\xi_2|$. The following result shows that this indeed the case.

Lemma 3.3. *For every $A \gg 1$ and $0 < \varepsilon \ll 1$, there exists an $R \gg A$ depending on A, ε, Ω, such that $Z_\Omega \cap S_{A,R}$ lies within a $O(\sqrt{\varepsilon})$ neighborhood of Z_Q, where $S_{A,R}$ is the slab*

$$S_{A,R} := \{(\xi_1, \xi_2) : |\xi_1| \geq R; |\xi_2| \leq A\}. \tag{3.7}$$

Proof. Fix A, ε. We may write

$$\hat{\chi}_\Omega = \hat{\chi}_{\Omega_-} + \hat{\chi}_Q + \hat{\chi}_{\Omega_+} \tag{3.8}$$

where Ω_- is the portion of Ω below $x_2 = -1/2$, and Ω_+ is the portion above $x_2 = 1/2$. In light of (3.4), it thus suffices to show that

$$|\hat{\chi}_{\Omega_\pm}(\xi_1, \xi_2)| \lesssim \varepsilon/|\xi_1| \tag{3.9}$$

on $S_{A,R}$. By symmetry it suffices to do this for Ω_+.

We may write Ω_+ as

$$\Omega_+ = \{(x, y) : -1/2 \leq x \leq 1/2; 1/2 \leq y \leq 1/2 + f(x)\} \tag{3.10}$$

where f is a concave function on $[-1/2, 1/2]$ such that $f(\pm 1/2) = 0$.

By continuity of f, we can find a $0 < \delta \ll \varepsilon$ such that

$$f(1/2 - \delta), f(\delta - 1/2) \leq \varepsilon. \tag{3.11}$$

Draw the line segment from $(1/2, 1/2)$ to $(1/2 - \delta, 1/2 + f(1/2 - \delta))$, and the line segment from $(-1/2, 1/2)$ to $(-1/2 + \delta, 1/2 + f(-1/2 + \delta)$. This divides Ω_+ into two small convex bodies and one large convex body. The diameter of the small convex bodies is $O(\varepsilon)$, and so their contribution to (3.9)is acceptable by Lemma 2.1. If R is sufficiently large depending on A, then (ξ_1, ξ_2) will always make an angle of $\gtrsim \delta/\varepsilon$ with the normals of the large convex body. By Lemma 2.2, the contribution of this large body is therefore $O(\delta/\varepsilon|\xi|^2)$, which is acceptable if R is sufficiently large.

Let $A \gg 1$ and $0 < \varepsilon \ll 1$, and let R be as in Lemma 3.3. Since $\Lambda - \Lambda \subset Z_\Omega$, then by Lemma 3.3 we see that $\Lambda \cap (\xi + S)$ lies in an $O(\sqrt{\varepsilon})$ neighborhood of $Z_Q + \xi$ for all $\xi \in \Lambda$. Suppose that we could find $\xi, \xi' \in \Lambda$ such that $|\xi - \xi'| \ll A$ and

$$\mathrm{dist}(\xi - \xi', G) \gg \sqrt{\varepsilon}. \tag{3.12}$$

It follows that

$$\Lambda \cap (\xi + S_{A,R}) \cap (\xi' + S_{A,R}) \tag{3.13}$$

lies in an $O(\sqrt{\varepsilon})$ neighborhood of $G + \xi$ and in an $O(\sqrt{\varepsilon})$ neighborhood of $G + \xi'$. Since Λ has separation $\gtrsim 1$, it follows that Λ has density at most $1 + O(1/A)$ in the set $(\xi +$

$S_{A,R}) \cap (\xi' + S_{A,R})$. However, this is a contradiction for A large enough since Λ needs to have asymptotic density $1/|\Omega| < 1/|Q| = 1$.

By letting $\varepsilon \to 0$ and $A \to \infty$ we see that

$$\xi - \xi' \in G \tag{3.14}$$

for all $\xi, \xi' \in \Lambda$. In particular, $\Lambda \subset G$since $(0,0) \in \Lambda$.

Now suppose for contradiction that (3.3) failed. Then there exists $(\xi_1, \xi_2) \in \Lambda$ such that $\xi_1 \notin \mathbf{Z}$. Since $\Lambda \subset G$, we thus have that $\xi_2 \in \mathbf{Z}$. From (3.14) we thus see that

$$\Lambda \subset \mathbf{R} \times \mathbf{Z}. \tag{3.15}$$

For each integer k, let R_k denote the intersection of Λ with $\mathbf{R} \times \{k\}$.

Let $A \gg 1$ and $0 < \varepsilon \ll 1$, and let R be as in Lemma 3.3. If $\xi, \xi' \in R_k$ and $|\xi - \xi'| \gg R$, then by Lemma 3.3 we see that $\xi - \xi'$ lies in a $O(\sqrt{\varepsilon})$ neighbourhood of $\mathbf{Z}$. From this and the separation of Λ we see that one has

$$\#\{(\xi_1, k) \in R_k : |\xi_1| \leq M\} \lesssim M + R \tag{3.16}$$

for all k and M. Summing this for $-M < k < M$ and then letting $M \to \infty$ we see that Λ has asymptotic density at most 1, a contradiction. This proves (3.3), and Proposition 3.1 is proved.

Proof of Proposition 3.2. By an affine rescaling we may assume that $x = e_1/2$ and $n = e_1$, so that our task is again to show (3.3). We shall prove the following analogue of Lemma 3.3.

Lemma 3.4. *For all $A \gg 1$, $0 < \varepsilon \ll 1$ there exists an $R \gg 1$ depending on A, ε, Ω such that $Z_\Omega \cap B(Re_1, A)$ lies within $O(\varepsilon)$ of $\mathbf{Z} \times \mathbf{R}$.*

Proof. Fix A, ε. We can write Ω as

$$\Omega = \{(x, y) : -1/2 \leq x \leq 1/2; -f(-x) \leq y \leq f(x)\} \tag{3.17}$$

where $f(x)$ is a concave function on $[-1/2, 1/2]$ which vanishes at the endpoints of this interval but is positive on the interior.

For each $0 < \delta \ll 1$, define

$$S(\delta) := \frac{f(1/2 - \delta) + f(-1/2 + \delta)}{\delta}. \tag{3.18}$$

The function $\delta S(\delta)$ is decreasing to 0 as $\delta \to 0$. Thus we may find a $0 < \delta_0 \ll \varepsilon/A$ such that $\delta_0 S(\delta_0) \lesssim \varepsilon/A$.

Fix δ_0, and let l_+, l_- be the line segments from $(1/2 - 2\delta_0, 0)$ to $(1/2 - \delta_0, f(1/2 - \delta_0))$ and $(1/2 - \delta_0, -f(-1/2 + \delta_0))$ respectively, and let $-l_+$, $-l_-$ be the reflections of these line segments through the origin.

By symmetry we have

$$\hat{\chi}_\Omega = 2\Re(\widehat{\chi}_{\Omega_+} + \widehat{\chi}_{\Gamma_0}) \tag{3.19}$$

where Ω_+ is the portion of Ω above l_+, $-l_-$, and the e_1 axis, and Γ_0 is the small portion of Ω between l_+ and l_-.

Since we are assuming Ω to have normal e_1 at $e_1/2$, we see that $S(\delta) \to \infty$ as $\delta \to 0$. Thus we may find a $0 < \delta \ll \delta_0$ such that

$$S(\delta) \gg 1 + \frac{1}{\varepsilon} S(\delta_0). \tag{3.20}$$

Fix this δ. By Corollary 2.4 we may find an $R \sim 1/\delta$ such that

$$|\hat{\chi}_{\Gamma_0}(Re_1)| \gtrsim (f(1/2-\delta) + f(-1/2+\delta))\delta = \delta^2 S(\delta). \tag{3.21}$$

Fix this R. Let m_+, m_- be the line segments from $(1/2 - 2\delta, 0)$ to $(1/2 - \delta, f(1/2 - \delta))$ and $(1/2 - \delta_0, -f(-1/2 + \delta))$ respectively. We can partition

$$\hat{\chi}_{\Gamma_0} = \hat{\chi}_{\Gamma_+} + \hat{\chi}_{\Gamma_-} + \hat{\chi}_{\Gamma} \tag{3.22}$$

where Γ_+ is the portion of Γ above m_+ and the e_1 axis, Γ_- is the portion below m_- and the e_1 axis, and Γ is the portion between m_+ and m_-.

The convex body $\Gamma - e_1/2$ is contained inside a ball of radius $O(S(\delta)\delta)$, hence by (0.2) we have

$$|\nabla \hat{\chi}_{\Gamma - e_1/2}(\xi)| \lesssim (\delta S(\delta))^2/R \lesssim (\delta_0 S(\delta_0))\delta^2 S(\delta) \lesssim \frac{\varepsilon}{A}\delta^2 S(\delta) \tag{3.23}$$

for $\xi \in B(Re_1, A)$.

If $\xi \in B(Re_1, A)$, then ξ makes an angle of

$$O(A/R) = O(A\delta) \ll O(\delta/(\delta_0 S(\delta_0))) = O(\delta/(\delta S(\delta))) = O(1/S(\delta)) \tag{3.24}$$

with the e_1 axis, and hence makes an angle of $\gtrsim 1/S(\delta)$ with the convex bodies $\Gamma_+ - e_1/2$, $\Gamma_- - e_1/2$. Since these bodies are in a ball of radius $O(S(\delta_0)\delta_0) = O(\varepsilon/A)$, we see from Lemma 2.2 that

$$|\nabla \hat{\chi}_{\Gamma_\pm - e_1/2}(\xi)| \lesssim \frac{\varepsilon}{A} S(\delta)/R^2 \sim \frac{\varepsilon}{A}\delta^2 S(\delta). \tag{3.25}$$

Summing, we obtain

$$|\nabla(e^{\pi i \xi_1}\hat{\chi}_{\Gamma_0}(\xi))| \lesssim \frac{\varepsilon}{A}\delta^2 S(\delta). \tag{3.26}$$

Integrating this and (3.21) we get

$$\hat{\chi}_\Gamma(\xi) = \hat{\chi}_{\Gamma_0}(Re_1)(e^{\pi i(R-\xi_1)} + O(\varepsilon)). \tag{3.27}$$

If $\xi \in B(Re_1, A)$, then ξ makes an angle of $\gtrsim 1/S(\delta_0)$ with every normal of Ω_+. From Lemma 2.2 we get

$$|\hat{\chi}_{\Omega_+}(\xi)| \lesssim S(\delta_0)/R^2 \sim S(\delta_0)\delta^2 \ll \varepsilon\delta^2 S(\delta) \tag{3.28}$$

on $B(Re_1, A)$. From this, (3.20), (3.22), and (3.23) we obtain

$$\hat{\chi}_\Omega(\xi) = 2\Re(\hat{\chi}_\Gamma(Re_1)(e^{\pi i(R-\xi_1)} + O(\varepsilon))) \tag{3.29}$$

on $B(Re_1, A)$, and the Lemma follows.

Let $A \gg 1$, $0 < \varepsilon \ll 1$, and let R be as in Lemma 3.4. If $\xi \in \Lambda$ are such that $|\xi| \ll A$, then from Lemma 3.4 we see that

$$Z_\Omega \cap B(Re_1 + \xi, A) \cap B(Re_1, A) \tag{3.30}$$

lies within $O(\varepsilon)$ of $(\mathbf{Z} \times \mathbf{R})$, and within $O(\varepsilon)$ of $(\mathbf{Z} \times \mathbf{R}) + \xi$. Since Z_Ω has asymptotic density $1/|\Omega|$, it has a non-empty intersection with $B(Re_1, A) \cap B(Re_1 + \xi, A)$, and thus ξ must lie within $O(\varepsilon)$ neighbourhood of $\mathbf{Z} \times \mathbf{R}$. Taking $\varepsilon \to 0$ and then $A \to \infty$ we obtain (3.3), and Proposition 3.2 is proved.

Conclusion of the argument

We now use Proposition 3.1 and 3.2 to show that the only convex symmetric bodies with spectra are the quadrilaterals and hexagons. We may assume of course that Ω is not a quadrilateral or a hexagon.

Suppose that there are two points x, x' in $\partial\Omega$ for which either Proposition 3.1 or Proposition 3.2 applies. From elementary geometry we thus see that Λ must live in a lattice of density $|2x \wedge 2x'|$. It follows that

$$4|x \wedge x'| \geq |\Omega| \tag{4.1}$$

for all such x, x'. Since $|x| \sim 1$ on $\partial\Omega$, this implies that there are only a finite number of x for which Proposition 3.1 and Proposition 3.2 applies. Since almost every point in $\partial\Omega$ has a unit normal, the only possibility left is that Ω is a polygon.

Label the vertices of Ω cyclically by $x_1, \ldots, x_{2n}$. Since Ω is not a quadrilateral or a hexagon, we have $n \geq 4$. By symmetry we have $x_{n+i} = -x_i$ for all i (here we use the convention that $x_{2n+i} = x_i$).

From Proposition 3.1 we have

$$\xi \cdot (x_i - x_{n+i-1}) \in \mathbf{Z} \tag{4.2}$$

for all $\xi \in \Lambda$. First suppose that n is even. Then $n-1$ is coprime to $2n$, and by repeated application of (4.2) we see that

$$\xi \cdot (x_i - x_j) \in \mathbf{Z} \tag{4.3}$$

for all i, j. Arguing as in the derivation of (4.1) we thus see that

$$|(x_i - x_j) \cdot (x_i - x_k)| \geq |\Omega| \tag{4.4}$$

for all i, j, k. In other words, the triangle with vertices x_i, x_j, x_k has area at least $|\Omega|/2$ for all i, j, k. But Ω can be decomposed into $2n-2$ such triangles, a contradiction since $n \geq 4$.

Now suppose that n is odd, so that $n \geq 5$. Then $n-1$ and $2n$ have the common factor of 2. Arguing as before we see that (4.3) holds for all i, j, k of the same parity. But Ω contains the three disjoint triangles with vertices (x_1, x_3, x_5), (x_1, x_5, x_7), and (x_1, x_7, x_9) respectively, and we have a contradiction.

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